Table 1 Variation of stress concentration

η =	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
$2p_{\xi\eta}/P\mu p_c kc$ =	0	0.34	0.36	0.30	0	−0.30	−0.36	−0.34	0

prevailing at infinity, one assumes

$$u_r = 0 \qquad u_\theta = \vartheta = \vartheta(r,z) \qquad u_z = 0 \tag{1}$$

The strain elements then are given by

$$\begin{aligned} e_{rr} = e_{\theta\theta} = e_{zz} = e_{rz} = 0 \\ e_{r\theta} = r(\partial/\partial r)(\vartheta/r) \qquad e_{\theta z} = r(\partial/\partial z)(\vartheta/r) \end{aligned} \tag{2}$$

From the preceding strain components, the stresses can be derived as follows:

$$\begin{aligned} p_{rr} = p_{\theta\theta} = p_{zz} = p_{rz} = 0 \\ p_{r\theta} = c_{66}r(\partial/\partial r)(\vartheta/r) \qquad p_{\theta z} = c_{44}r(\partial/\partial z)(\vartheta/r) \end{aligned} \tag{3}$$

Two of the body stress equations of equilibrium are seen to be satisfied identically, and the third one becomes

$$(\partial p_{r\theta}/\partial r) + (\partial p_{\theta z}/\partial z) + 2(p_{r\theta}/r) = 0$$

Substituting the values of the stress components from Eq. (3), one gets

$$(\partial^2\vartheta/\partial r^2) + (1/r)(\partial\vartheta/\partial r) - (\vartheta/r^2) + k^2(\partial^2\vartheta/\partial z^2) = 0 \tag{4}$$

where $k^2 = c_{44}/c_{66}$. The inclusion is an oblate spheroid whose boundary is given by

$$(r^2/a^2) + (z^2/b^2) = 1 \qquad a > b \tag{5}$$

The inclusion is supposed to be fixed rigidly. Put $z = kz'$ in Eq. (4), which then reduces to

$$(\partial^2\vartheta/\partial r^2) + (1/r)(\partial\vartheta/\partial r) - (\vartheta/r^2) + (\partial^2\vartheta/\partial z'^2) = 0 \tag{6}$$

The boundary becomes

$$(r^2/a^2) + [(z')^2/(b^2/k^2)] = 1$$

Assuming $k > 1$, it will be found that $a^2 > b^2/k^2$. At a great distance from the inclusion it is supposed that $\vartheta(= \vartheta_1)$ is due to torsion of the circular cylinder only. That is, at a great distance from the inclusion, $\vartheta_1 = p_c rz = p_c krz'$, p_c being the twist.

In the case of an oblate spheroid, introduce the transformation given by

$$r + iz' = c\cosh(\xi + i\eta)$$

so that

$$\begin{aligned} r &= c\cosh\xi\cos\eta \\ z' &= c\sinh\xi\sin\eta \\ 1/h^2 &= c^2(\cosh^2\xi - \cos^2\eta) \end{aligned}$$

If one puts

$$\begin{aligned} a = c\cosh\alpha \qquad & b/k = c\sinh\alpha \\ c = [a^2 - (b^2/k^2)]^{1/2} \qquad & \tanh\alpha = b/ka \end{aligned}$$

one has $\xi = \alpha$ when r and z' are connected by (7). Equation (6) now becomes

$$(\partial/\partial\xi)[(\partial\vartheta/\partial\xi) + \vartheta\tanh\xi] + (\partial/\partial\eta)[(\partial\vartheta/\partial\eta) - \vartheta\tan\eta] = 0$$

Take as a solution

$$\vartheta = \{A(d/d\xi)[Q_2(i\sinh\xi)]\}(d/\eta)P_2(\sin\eta)$$

where $P_2(\sin\eta)$ and $Q_2(i\sinh\xi)$ are Legendre functions of the first and second kind, respectively. For large ξ, $\vartheta_1 = p_c kc^2 \sinh\xi\cosh\xi\sin\eta\cos\eta$.

Now, the only boundary condition to be satisfied is $\vartheta + \vartheta_1 = 0$ on $\xi = \alpha$, which gives

$$A = -(p_c kc^2/3)\{\sinh\alpha\cosh\alpha/(d/d\alpha)[Q_2(i\sinh\alpha)]\}$$

Hence the total displacement is given by

$$\begin{aligned} \vartheta_2 &= \vartheta + \vartheta_1 \\ &= \frac{p_c kc^2}{3}\left\{\sinh\xi\cosh\xi - \frac{\sinh\alpha\cosh\alpha}{(d/d\alpha)[Q_2(i\sinh\alpha)]}\frac{d}{d\xi} \times [Q_2(i\sinh\xi)]\right\}\frac{d}{d\eta}P_2(\sin\eta) \end{aligned}$$

On $\xi = \alpha$, $p_{\xi\xi} = p_{\eta\eta} = 0$, while

$$[p_{\xi\eta}] = \frac{\mu p_c kc}{6}\frac{(d/d\eta)P_2(\sin\eta)}{(\cosh^2\alpha - \cos^2\eta)^{1/2}} \times \left\{\frac{2(d/d\alpha)[Q_2(i\sinh\alpha)]\cosh 2\alpha - \sinh 2\alpha(d^2/d\alpha^2)[Q_2(i\sinh\alpha)]}{(d/d\alpha)[Q_2(i\sinh\alpha)]}\right\}$$

To have an idea of the variation of stress around the inclusion, assume, for example, $\alpha = 1$. Then

$$p_{\xi\eta} = \frac{P\mu P_c k_c}{2}\frac{\sin\eta\cos\eta}{(2.3811 - \cos^2\eta)^{1/2}}$$

where

$$P = \left\{\frac{2(d/d\alpha)[Q_2(i\sinh\alpha)]\cosh 2\alpha - \sinh 2\alpha(d^2/d\alpha^2)[Q_2(i\sinh\alpha)]}{(d/d\alpha)[Q_2(i\sinh\alpha)]}\right\}$$

Variation of $p_{\xi\eta}$ on the inclusion for different values of η is shown in Table 1.

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Stagnation Point Heat Transfer of a Blunt-Nosed Body in Low-Density Flow

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A SHOCK-boundary-layer matching scheme has been introduced in the study of stagnation point heat transfer characteristics of a blunt-nosed body in hypersonic flow where the boundary layer thickness is of the order of the detachment distance. The velocity components, stress components, temperature, heat flux, pressure, and density are matched on a matching surface, which can be determined uniquely from the analysis. The heat transfer results merge with those developed in Ref. 1 for large Reynolds number and show smooth transition from high Reynolds number to Reynolds number of the order of 50 [R_{es}=

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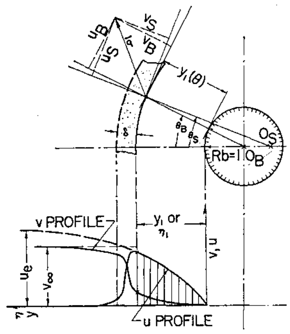

Fig. 1

$\rho s(h_{s\infty}R_b)^{1/2}/\mu_s]$, where the results approach those obtained from the free-molecular theory.

Assuming (Fig. 1) that both the shock thickness δ and the boundary layer thickness y_1 are much less than the nose radius $R_b = 1$ (here one has a cold blunt-nosed body with a hypersonic flight speed in low-density field), the usual shock conservation equations[2] can be obtained by integrating the approximated system of Navier-Stokes equations valid across the shock thickness [note that $\partial/\partial r_s \sim R_e(\partial/\partial\theta_s)$ is used in the order-of-magnitude analysis, where R_e is based on the shock radius of curvature] along a ray of constant θ_s up to the matching point y_1, which is as yet to be determined. This system of four equations, namely, the continuity equation, the circumferential and radial component of momentum equations, and the energy equations, readily can be transformed and expressed in the curvilinear coordinate system with the body surface as one of the coordinate surfaces. Close to the axis of symmetry, this system coincides with the spherical polar coordinate system with O_B, the center of curvature of the shock surface on the axis, as the origin, and the distance between the centers of curvature of the body and the shock $e = \overline{O_SO_B}$ enters as a parameter [note that if, one treats $y_1 = y_1(\theta_B)$ as a variable instead of e, then $(d^2y_1/d\theta_B^2)\theta_B = 0$ appears]. Referring to this system (r_B, θ_B), a typical equation, for instance, the radial component of momentum equation, is of the form

$$\left\{\frac{\Gamma - 1}{\Gamma}\bar{\rho}[\bar{h}_s - \tfrac{1}{2}(\bar{v}^2 + \bar{u}^2)] - \bar{p}_\infty\right\}\{e^2 + (1 + y_1)^2 + 2e(1 + y_1)\cos\theta_B\} = [e + (1 + y_1)\cos\theta_B]^2 + [e + (1 + y_1)\cos\theta_B]\{(1 + y_1 + e\cos\theta_B)\bar{v} - (e\sin\theta_B)\bar{u}\} + \epsilon_2\left\{\left(\frac{e\sin\theta_B}{1 + y_1}\right)[(e\sin\theta_B)\bar{v} + (1 + y_1 - e\cos\theta_B)\bar{u}] + \frac{\partial\bar{v}}{\partial y}(1 + y_1 + e\cos\theta_B)^2 - \frac{\partial\bar{u}}{\partial y}(e\sin\theta_B)(1 + y_1 + e\cos\theta_B) + \frac{1}{1 + y_1}\frac{\partial\bar{u}}{\partial\theta_B}(e^2\sin^2\theta_B) - \frac{1}{1 + y_1}\frac{\partial\bar{v}}{\partial\theta_B}(e\sin\theta_B) \times (1 + y_1 + e\cos\theta_B)\right\}$$

where all the barred quantities are normalized with respect to the freestream values, and $\Gamma = c_p/c_v$ is evaluated at y_1; $\epsilon_2 = (\mu' + \frac{4}{3}\mu)/\rho_\infty V R_b$. The variables involved in the equations are $\bar{\rho}$, $\bar{T}$, $\bar{u}$, $\bar{v}$, $\partial\bar{u}/\partial y$, $\partial\bar{v}/\partial y$, $\partial\bar{T}/\partial y$, and y_1. Since they are functions of θ_B, the $\partial\bar{u}/\partial\theta_B$, $\partial\bar{v}/\partial\theta_B$, etc., do not appear as extra variables. This system of conservation equations now can serve as the outer boundary conditions for which the solution of the boundary layer equations is sought. This can be done numerically by machine integration after a suitable transformation of the variables. However, by making use of the similarity solution, the problem can be solved more easily, following the same type of approach as employed in Ref. 1; namely, the solution of boundary layer equation or the flow field in the boundary layer is defined in terms of the fictitious outer quantities $\bar{u}_e$, $\bar{\rho}_e$, $\bar{h}_{s_e}$ and the two space variables y and θ, or the transformed variables η and ξ. The four boundary conditions or the four shock conservation equations at y_1 or η_1, together with the transformation between η and y,

$$\eta_1 = \frac{\rho_e u_e}{(2\xi)^{1/2}}\int_0^{y_1} R_b^2 \sin\theta_B \left(\frac{\rho}{\rho_e}\right) dy = F(\bar{\rho}_e, \bar{u}_e, y_1)$$

form the five equations for the determination of the five unknowns $\bar{\rho}_e(\theta_B)$, $\bar{u}_e(\theta_B)$, $\bar{h}_{s_e}(\theta_B)$, $\eta_1(\theta_B)$, and $y_1(\theta_B)$. For the five equations to hold near the axis of symmetry, an expansion scheme of the type $\bar{\rho}_e(\theta_B) = {}_0\bar{\rho}_e + {}_2\bar{\rho}_e\theta_B^2$, etc., around $\theta_B = 0$ for all the variables can be introduced, and, by collecting terms of equal powers of θ_B^0 and θ_B^2, one gets 10 algebraic equations for the 10 unknown coefficients, ${}_0\bar{\rho}_e$, ${}_2\bar{\rho}_e$, ${}_1\bar{u}_e$, ${}_3\bar{u}_e$, ${}_0\bar{h}_{se}$, ${}_2\bar{h}_{se}$, ${}_0\eta_1$, ${}_2\eta_1$, ${}_0y_1$, and ${}_2y_1$ (or e). The equations are composed of two groups of five equations. One group involves only e and the coefficients ${}_1\bar{u}_e$, ${}_0\bar{\rho}_e$, ${}_0\bar{h}_{se}$, ${}_0\eta_1$, ${}_0y_1$ as unknowns. For a given value of e, the remaining five coefficients can be determined by solving the equations on the axis ($\theta_B = 0$). The other group of equations involves all the 10 coefficients but is linear in ${}_3\bar{u}_e$, ${}_2\bar{h}_{se}$, ${}_2\bar{\rho}_e$, and ${}_2\eta_1$. An iteration procedure with an initial value of e will determine the complete set of coefficients with a new value of e for the next cycle of iteration.

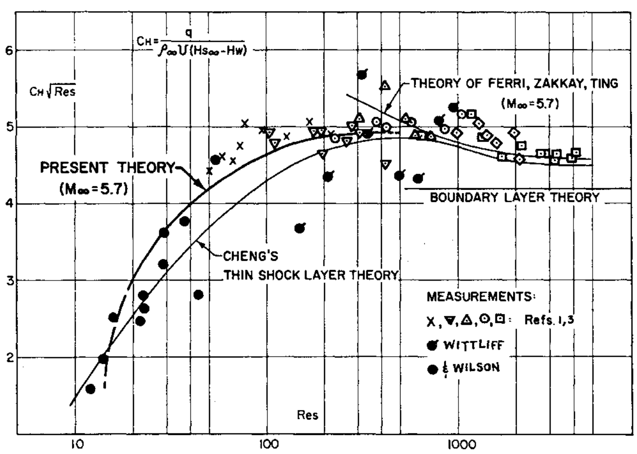

Fig. 2 Heat transfer value $C_H(R_{es})^{1/2}$ vs R_{es}

With $M_\infty = 5.7$ and T_0 = the freestream stagnation temperature = 2100°R, calculation based on a linear μ, T relation has been performed. Result of heat transfer value $C_H (R_{es})^{1/2} = q(R_{es})^{1/2}/\rho_\infty U(H_{s\infty} - H_w)$ is shown in Fig. 2 as comparing with Cheng's thin shock layer theory and also the experimental data of Wittliff and Wilson.[5,6] The details of the scheme will be presented in a forthcoming report.[4]

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Vibrations of Skew Cantilever Plates

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References 1 and 2 describe the results of calculations of frequencies and nodal lines of vibrating rectangular cantilever plates. These calculations have been extended to vibrating skew cantilever plates. An IBM Fortran program is available. The plate is assumed to be vibrating transversely, in a single harmonic. Figure 1 gives the geometry of the plate.

In Refs. 1 and 2, the solution was obtained by a Fourier-sine-series method. Here, it was found convenient to use the Rayleigh-Ritz method. The mathematical development is the same as that of Ref. 3. The result of applying the Rayleigh-Ritz method is an infinite, real, symmetric matrix, for which the eigenvalues and eigenvectors are to be calculated. In the calculations the matrix is truncated in the usual fashion to successive finite-order matrices, and the limits of the eigenvalues and eigenvectors are evaluated numerically. The method used for calculating the eigenvalues and eigenvectors of the finite matrices is developed in Ref. 4.

In order to obtain any degree of accuracy in calculating the eigenvectors, and therefore the nodal lines, it has been found absolutely essential that the same finite-order matrix be used as was used for the eigenvalues. In fact, it is desirable to calculate the eigenvalues to several more significant figures than the number required for the nodal lines. This is in direct contrast to the Fourier-series method described in Refs. 1 and 2, where an estimated limit for the frequencies was used in calculating the nodal lines.

References 1 and 2 describe the variations of the frequencies and the nodal lines as functions of the ratio of sides a/b for a rectangular plate. Thus, the different harmonics were thought of as "frequency curves." It is now possible to consider the frequencies and nodal lines as functions of two independent variables, a/b and θ. Instead of referring to "frequency curves," it is now possible to refer to "frequency sheets."

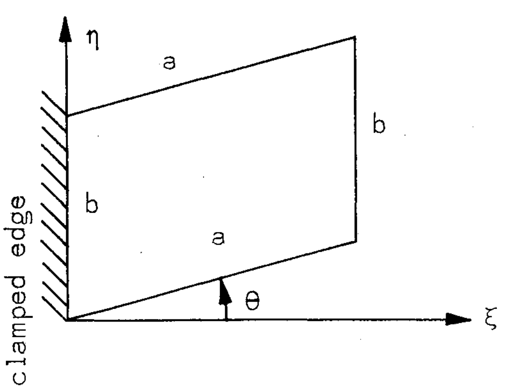

Fig. 1 Geometry of the plate

References 1 and 2 referred to "transition points," points at which the frequency curves should have crossed each other but actually refused to do so, markedly changing their curvature instead. It is now possible to state the existence of "transition curves," curves along which the frequency sheets refuse to cross each other but instead markedly change their curvature. However, along different segments of a transition curve a wide variation is possible in the distance between two frequency sheets. In fact, the sheets can actually touch (become tangent to) each other at isolated points.

As in Refs. 1 and 2, the nodal lines rotate about one or several points as the frequency sheets change their curvature. Reference 5 gives a detailed description of the mathematical development and the results, as well as the Fortran program used in calculating the eigenvalues and eigenvectors.

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An Example of Boundary Layer Formation

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A major difficulty in the teaching of fluid dynamics is the lack of a simple exact solution of the Navier-Stokes equations in which both the viscous and inertial forces are active. Viscous forces only are involved in Poiseuille and Couette flows, and consequently the velocity fields are independent of the Reynolds number. There are two exact solutions that depend on viscous and inertial forces, namely the von Kármán flow produced by a rotating disk and the Jeffrey-Hamel flow in a converging or diverging channel. Interesting as these solutions are, they suffer from the disadvantage of requiring the solution of nonlinear differential equations, and the velocity fields cannot be expressed in simple terms. For teaching purposes, a solution is required which can be expressed in simple functions, is exact, and involves a balance between viscous and inertial forces, so that the dependence on the Reynolds number can be exhibited and the formation of a boundary layer as the Reynolds number increases demonstrated. It also would be helpful if a class of solutions rather than a single solution were known, to provide examples for the student to find for himself. It does not seem to be recognized widely that a class of solutions satisfying these requirements exists, a particular case

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